

RECURSIVE FORMULAS FOR SUMS OF SQUARES AND SUMS OF TRIANGULAR NUMBERS

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ABSTRACT. We prove recursive formulas for sums of squares and sums of triangular numbers in terms of sums of divisors functions and we give a variety of consequences of these formulas. Intermediate applications include statements about positivity of the coefficients of some infinite products.

1. INTRODUCTION

The purpose of this paper is to give inductive formulas for sums of squares, sums of triangular numbers, and mixed sums of squares and triangular numbers. Our main tool is the following theorem. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Theorem 1. [3] *Let $\alpha \in \mathbb{N}$. Let $A_1, A_2, \dots, A_\alpha \subseteq \mathbb{N}$ and $\underline{A} = (A_1, A_2, \dots, A_\alpha)$ and let $f_i : A_i \rightarrow \mathbb{C}$ for $i = 1, 2, \dots, \alpha$ be arithmetic functions and let $\underline{f} = (f_1, f_2, \dots, f_\alpha)$. If both*

$$F_{\underline{A}}(x) = \prod_{i=1}^{\alpha} \prod_{n \in A_i} (1 - x^n)^{-\frac{f_i(n)}{n}} = \sum_{n=0}^{\infty} p_{\underline{A}, \underline{f}}(n) x^n$$

and

$$\sum_{i=1}^{\alpha} \sum_{n \in A_i} \frac{f_i(n)}{n} x^n$$

converge absolutely and represent analytic functions in the unit disk $|x| < 1$, then

$$np_{\underline{A}, \underline{f}}(n) = \sum_{k=1}^n \left(p_{\underline{A}, \underline{f}}(n-k) \sum_{i=1}^{\alpha} f_{i, A_i}(k) \right),$$

where $p_{\underline{A}, \underline{f}}(0) = 1$ and

$$f_{i, A_i}(k) = \sum_{\substack{d|k \\ d \in A_i}} f_i(d).$$

We note that Theorem 1 for the special case $\alpha = 1$, has been given in Apostol [1] and in Robbins in [6] to give formulas relating arithmetic functions to sums of divisors functions. The authors' key argument is that generating functions for the arithmetic functions they considered have the form of infinite products ranging over a single set of natural numbers. In our previous work [3] we used Theorem 1 to deal with arithmetic functions whose generating functions involve finitely many infinite

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products ranging over different sets of natural numbers. We derived a variety of inductive formulas for such functions. In the present note we shall continue employing Theorem 1 to deduce inductive formulas for sums of squares, sums of triangular numbers, and mixed sums of squares and triangular numbers, see Section 2. We notice at this point that there is a large literature on sums of squares and sums of triangular numbers. The interested reader is referred to the classical volumes of Dickson [2] and the recent book of Williams [7] along with their references. With an essential help of identities given in Williams in [7], we will give a variety of consequences of our recursive formulas, see Section 3. Further, we shall prove that the coefficients of some infinite products are all positive, see Section 4. By way of example, we will prove the following results:

(1)

$$\left(\sum_{n=1}^{\infty} (\sigma(n) - 4\sigma(n/4)) x^n \right) \left(\sum_{n=1}^{\infty} (\sigma^*(n) - 4\sigma^*(n/2)) x^n \right) = \sum_{n=1}^{\infty} (n(\sigma(n) - 4\sigma(n/4)) - (\sigma^*(n) - 4\sigma^*(n/2))) x^n,$$

see Corollary 1 below. Here $\sigma(n)$ and $\sigma^*(n)$ are as in Definitions 1 and 3 below.

(2) If $p \equiv 1 \pmod{4}$ is prime, then

$$\sum_{j=1}^{p-1} r_2(j) \left(\sigma^*(p-j) - 4\sigma^*\left(\frac{p-j}{2}\right) \right) = p-1,$$

refer to Theorem 6 below. Here $r_2(n)$ is the number of representations of n as a sum of two squares.

(3) If p and $4p+1$ are primes, then

$$\sum_{j=1}^{p-1} t_2(j) \left(\sigma(p-j) - 4\sigma\left(\frac{p-j}{2}\right) \right) = -1,$$

see Theorem 8 below. Here $t_2(n)$ is the number of representations of n as a sum of two triangular numbers.

(4) The following is a direct consequence of Theorem 11 below. If $x \in \mathbb{C}$ with $|x| < 1$ and $a \in \mathbb{N}$, then the coefficients of the infinite series

$$\prod_{n=1}^{\infty} (1-x^n)^{-a} (1-x^{3n})^a, \quad \prod_{n=1}^{\infty} (1-x^n)^{-a} (1-x^{3n-1})^a,$$

and

$$\prod_{n=1}^{\infty} (1-x^n)^{-a} (1-x^{3n})^a (1-x^{3n-1})^a$$

are all positive.

Our identities involve sums of divisors functions which are introduced in the following three definitions.

Definition 1. Let the function σ be defined on \mathbb{Q} as follows: $\sigma(0) = 1$, if $q \in \mathbb{Q} \setminus \mathbb{N}_0$, then $\sigma(q) = 0$, and if $q \in \mathbb{N}$, then

$$\sigma(q) = \sum_{d|q} d.$$

Definition 2. Let $n, r \in \mathbb{N}_0$, let $m \in \mathbb{N}$, and let

$$\sigma_{r,m}(n) = \sum_{\substack{d|n \\ d \equiv r \pmod{m}}} d$$

If $m = 2$ and $r = 1$ we often write $\sigma^o(n)$ rather than $\sigma_{1,2}(n)$ and if $m = 2$ and $r = 0$ we often write $\sigma^E(n)$ rather than $\sigma_{0,2}(n)$.

We have the following basic facts: for $m, n \in \mathbb{N}$

$$(1) \quad \sigma(n) = \sigma^o(n) + \sigma^E(n), \quad \sigma_{m,2m}(n) = m\sigma^o(n/m), \quad \text{and} \quad \sigma_{0,m}(n) = m\sigma(n/m).$$

Definition 3. Let the function σ^* be defined on \mathbb{Q} as follows: if $q \in \mathbb{Q} \setminus \mathbb{N}$, then $\sigma^*(q) = 0$ and if $q \in \mathbb{N}$, then

$$\sigma^*(q) = \sum_{d|q, \frac{q}{d} \text{ odd}} d.$$

By [7, Theorem 3.4] we have for all $n \in \mathbb{N}$ that

$$(2) \quad \sigma^*(n) = \sigma(n) - \sigma(n/2).$$

Further we will need the following two identities due to Jacobi and Gauss respectively:

$$(3) \quad \prod_{n=1}^{\infty} (1 - x^{2n})(1 + x^{2n-1})^2 = \sum_{n=-\infty}^{\infty} x^{n^2}.$$

$$(4) \quad \prod_{n=1}^{\infty} (1 - x^{2n})(1 - x^{2n-1})^{-1} = \sum_{n=0}^{\infty} x^{\frac{n(n+1)}{2}}.$$

Identities (3) and (4) can be found for instance in Hardy and Wright [4].

2. GENERAL FORMULAS

In this section we give inductive formulas for sums of squares, sums of triangular numbers, and mixed sums of squares and triangular numbers.

Definition 4. Let $k \in \mathbb{N}_0$ and let the function $r_k(n)$ be defined on \mathbb{N}_0 as follows:

$$r_k(n) = \#\{(x_1, x_2, \dots, x_k) \in \mathbb{Z}^k : n = x_1^2 + x_2^2 + \dots + x_k^2\}.$$

Theorem 2. If $n \in \mathbb{N}$, then

$$nr_k(n) = 2k \left(\sigma^*(n) - 4\sigma^*(n/2) + \sum_{j=1}^{n-1} r_k(j) \left(\sigma^*(n-j) - 4\sigma^*\left(\frac{n-j}{2}\right) \right) \right).$$

Proof. Taking k powers in identity (3) we have

$$\prod_{n=1}^{\infty} (1 - x^{2n})^k (1 + x^{2n-1})^{2k} = \left(\sum_{n=-\infty}^{\infty} x^{n^2} \right)^k = 1 + \sum_{n=1}^{\infty} r_k(n) x^n,$$

or equivalently,

$$\prod_{n=1}^{\infty} (1 - x^{2n})^k (1 - x^{4n-2})^{2k} (1 - x^{2n-1})^{-2k} = 1 + \sum_{n=1}^{\infty} r_k(n) x^n.$$

Let in Theorem 1, $A_1 = 2\mathbb{N}$ and $f_1(n) = -kn$, $A_2 = 4\mathbb{N} - 2$ and $f_2(n) = -2kn$, and $A_3 = 2\mathbb{N} - 1$ and $f_3(n) = 2kn$. Then for $n \geq 1$

$$(5) \quad nr_k(n) = -k\sigma^E(n) - 2k\sigma_{2,4}(n) + 2k\sigma^o(n) + \sum_{j=1}^{n-1} r_k(j) (-k\sigma^E(n-j) - 2k\sigma_{2,4}(n-j) + 2k\sigma^o(n-j)).$$

Identities (1) and formula (2) yield

$$\begin{aligned} -\sigma^E(n) - 2\sigma_{2,4}(n) + 2\sigma^o(n) &= 2(\sigma(n) - \sigma(n/2)) - 8(\sigma(n/2) - \sigma(n/4)) \\ &= 2\sigma^*(n) - 8\sigma^*(n/2). \end{aligned}$$

Now put in formula (5) to conclude the desired identity. \square

Definition 5. Let $k \in \mathbb{N}_0$ and let the function t_k be defined on \mathbb{N}_0 as follows:

$$t_k(n) = \# \left\{ (x_1, x_2, \dots, x_k) \in \mathbb{N}_0^k : n = \sum_{i=1}^k \frac{x_i(x_i + 1)}{2} \right\}.$$

Theorem 3. If $n \in \mathbb{N}_0$, then

$$nt_k(n) = k \sum_{j=0}^{n-1} t_k(j) \left(\sigma(n-j) - 4\sigma\left(\frac{n-j}{2}\right) \right).$$

Proof. Taking k powers in identity (4) we have

$$\prod_{n=1}^{\infty} (1 - x^{2n})^k (1 - x^{2n-1})^{-k} = \left(\sum_{n=0}^{\infty} x^{\frac{n(n+1)}{2}} \right)^k = \sum_{n=0}^{\infty} t_k(n) x^n,$$

and the result follows by Theorem 1 applied to $A_1 = 2\mathbb{N}$, $f_1(n) = -kn$, $A_2 = 2\mathbb{N} - 1$, and $f_2(n) = kn$. \square

Definition 6. Let $k, l \in \mathbb{N}$ and let the function $u_{k,l}$ be defined on \mathbb{N}_0 as follows:

$$u_{k,l}(n) = \# \left\{ (x_1, \dots, x_k, y_1, \dots, y_l) \in \mathbb{Z}^k \times \mathbb{N}_0^l : n = \sum_{i=1}^k x_i^2 + \sum_{i=1}^l \frac{y_i(y_i + 1)}{2} \right\}.$$

Theorem 4. If $k, l, n \in \mathbb{N}$, then

$$\begin{aligned} nu_{k,l}(n) &= (2k+l)\sigma(n) - 2(5k+2l)\sigma(n/2) + 8k\sigma(n/4) + \\ &\quad \sum_{j=1}^{n-1} u_{k,l}(j) \left((2k+l)\sigma(n-j) - 2(5k+2l)\sigma\left(\frac{n-j}{2}\right) + 8k\sigma\left(\frac{n-j}{4}\right) \right). \end{aligned}$$

Proof. Clearly

$$\prod_{n=1}^{\infty} (1 - x^{2n})^{k+l} (1 + x^{2n-1})^{2k} (1 - x^{2n-1})^{-l} = 1 + \sum_{n=1}^{\infty} u_{k,l}(n) x^n,$$

that is,

$$\prod_{n=1}^{\infty} (1 - x^{2n})^{k+l} (1 - x^{4n-2})^{2k} (1 - x^{2n-1})^{-2k} (1 - x^{2n-1})^{-l} = 1 + \sum_{n=1}^{\infty} u_{k,l}(n) x^n,$$

equivalently,

$$\prod_{n=1}^{\infty} (1 - x^{2n})^{3k+l} (1 - x^{4n})^{-2k} (1 - x^{2n-1})^{-2k-l} = 1 + \sum_{n=1}^{\infty} u_{k,l}(n) x^n,$$

which by Theorem 1 gives the desired identity. \square

3. SUMS OF TWO, FOUR, AND EIGHT SQUARES

The following formulas for $r_2(n)$, $r_4(n)$, and $r_8(n)$ are well-known and can be found for instance in Williams [7]. If $n \in \mathbb{N}$, then

$$(6) \quad \begin{aligned} r_2(n) &= 4 \sum_{d|n} \left(\frac{-4}{d} \right), \\ r_4(n) &= 8\sigma(n) - 32\sigma(n/4), \\ r_8(n) &= 16(-1)^n \sum_{d|n} (-1)^d d^3, \end{aligned}$$

where

$$\left(\frac{-4}{d} \right) = \begin{cases} 1, & \text{if } d \equiv 1 \pmod{4}, \\ -1, & \text{if } d \equiv 3 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases}$$

We start by evaluating a convolution.

Theorem 5. *If $n \in \mathbb{N}$, then*

$$\begin{aligned} \sum_{j=1}^{n-1} \left(\sigma(j) - 4\sigma\left(\frac{j}{4}\right) \right) \left(\sigma^*(n-j) - 4\sigma^*\left(\frac{n-j}{2}\right) \right) = \\ n \left(\sigma(n) - 4\sigma\left(\frac{n}{4}\right) \right) - \left(\sigma^*(n) - 4\sigma^*\left(\frac{n}{2}\right) \right). \end{aligned}$$

Proof. By Theorem 2

$$(7) \quad nr_4(n) = 8\sigma^*(n) - 32\sigma^*(n/2) + \sum_{j=1}^{n-1} r_4(j) \left(8\sigma^*(n-j) - 32\sigma^*\left(\frac{n-j}{2}\right) \right),$$

which combined with (6) for $r_4(n)$ yields the result. \square

Corollary 1.

$$\begin{aligned} \left(\sum_{n=1}^{\infty} (\sigma(n) - 4\sigma(n/4)) x^n \right) \left(\sum_{n=1}^{\infty} (\sigma^*(n) - 4\sigma^*(n/2)) x^n \right) = \\ \sum_{n=1}^{\infty} (n\sigma(n) - 4n\sigma(n/4) - \sigma^*(n) + 4\sigma^*(n/2)) x^n. \end{aligned}$$

Proof. Immediate from Theorem 5. \square

Theorem 6. *Let p be prime.*

(a) *If $p \equiv 1 \pmod{4}$, then*

$$\sum_{j=1}^{p-1} r_2(j) \left(\sigma^*(p-j) - 4\sigma^*\left(\frac{p-j}{2}\right) \right) = p-1.$$

(b) If $p \equiv 3 \pmod{4}$, then

$$\sum_{j=1}^{p-1} r_2(j) \left(\sigma^*(p-j) - 4\sigma^*\left(\frac{p-j}{2}\right) \right) = -p-1.$$

Proof. by Theorem 2 we have

$$(8) \quad nr_2(n) = 4\sigma^*(n) - 16\sigma^*(n/2) + \sum_{j=1}^{n-1} r_2(j) \left(4\sigma^*(n-j) - 16\sigma^*\left(\frac{n-j}{2}\right) \right).$$

Further, if p is prime, then clearly $\sigma^*(p) = 1+p$ and $\sigma^*(p/2) = 0$. Moreover, by virtue of formulas (6) we have that $r_2(p) = 0$ for $p \equiv 3 \pmod{4}$ and $r_2(p) = 8$ for $p \equiv 1 \pmod{4}$. Now combine these facts with identity (8) to conclude parts (a) and (b). \square

Corollary 2. *Let $(p, p+2)$ be a twin-prime.*

(a) If $p \equiv 1 \pmod{4}$, then

$$\begin{aligned} \sum_{j=1}^{p+1} r_2(j) \left(\sigma^*(p+2-j) - 4\sigma^*\left(\frac{p+2-j}{2}\right) \right) = \\ -4 - \sum_{j=1}^{p-1} r_2(j) \left(\sigma^*(p-j) - 4\sigma^*\left(\frac{p-j}{2}\right) \right). \end{aligned}$$

(b) If $p \equiv 3 \pmod{4}$, then

$$\begin{aligned} \sum_{j=1}^{p+1} r_2(j) \left(\sigma^*(p+2-j) - 4\sigma^*\left(\frac{p+2-j}{2}\right) \right) = \\ - \sum_{j=1}^{p-1} r_2(j) \left(\sigma^*(p-j) - 4\sigma^*\left(\frac{p-j}{2}\right) \right). \end{aligned}$$

Proof. Immediate from Theorem 6. \square

Theorem 7. *Let p be an odd prime. Then*

$$\begin{aligned} (a) \quad \sum_{j=1}^{p-1} r_4(j) \left(\sigma^*(p-j) - 4\sigma^*\left(\frac{p-j}{2}\right) \right) &= p^2 - 1. \\ (b) \quad \sum_{j=1}^{p-1} r_8(j) \left(\sigma^*(p-j) - 4\sigma^*\left(\frac{p-j}{2}\right) \right) &= p^4 - 1. \end{aligned}$$

Proof. (a) As mentioned before $\sigma^*(p) = 1+p$ and $\sigma^*(p/2) = 0$. By identity (6) we have $r_4(p) = 8(1+p)$. Putting in formula (7) we get

$$8p(1+p) = 8(p+1) + \sum_{j=1}^{p-1} r_4(j) \left(8\sigma^*(p-j) - 32\sigma^*\left(\frac{p-j}{2}\right) \right),$$

which gives the desired identity.

(b) By Theorem 2

$$(9) \quad nr_8(n) = 16 \left(\sigma^*(n) - 4\sigma^*(n/2) + \sum_{j=1}^{n-1} r_8(j) \left(\sigma^*(n-j) - 4\sigma^*\left(\frac{n-j}{2}\right) \right) \right).$$

If p is an odd prime, then by formulas (6) we have $r_8(p) = 16(1 + p^3)$, which combined with identity (9) gives the result. \square

Corollary 3. *Let p be an odd prime. Then*

$$\begin{aligned} (a) \quad & \left(1 + \sum_{j=1}^{p-1} r_2(j) \left(\sigma^*(p-j) - 4\sigma^*\left(\frac{p-j}{2}\right) \right) \right)^2 = \\ & 1 + \sum_{j=1}^{p-1} r_4(j) \left(\sigma^*(p-j) - 4\sigma^*\left(\frac{p-j}{2}\right) \right). \\ (b) \quad & \left(1 + \sum_{j=1}^{p-1} r_4(j) \left(\sigma^*(p-j) - 4\sigma^*\left(\frac{p-j}{2}\right) \right) \right)^2 = \\ & 1 + \sum_{j=1}^{p-1} r_8(j) \left(\sigma^*(p-j) - 4\sigma^*\left(\frac{p-j}{2}\right) \right). \end{aligned}$$

Proof. Immediate from Theorems 6 and 7. \square

4. SUMS OF TWO, FOUR, AND SIX TRIANGULAR NUMBERS

The following formula for $t_2(n)$, $t_4(n)$, and $t_6(n)$ can be found in Ono, Robins, and Wahl [5] and in Williams [7].

$$\begin{aligned} (10) \quad & t_2(n) = \sum_{d|4n+1} \left(\frac{-4}{d} \right), \\ & t_4(n) = \sigma(2n+1), \\ & t_6(n) = \frac{-1}{8} \sum_{d|4n+3} \left(\frac{-4}{d} \right) d^2. \end{aligned}$$

Theorem 8. *If p and $4p+1$ are prime numbers, then*

$$\sum_{j=1}^{p-1} t_2(j) \left(\sigma(p-j) - 4\sigma\left(\frac{p-j}{2}\right) \right) = -1.$$

Proof. Suppose that p and $4p+1$ are primes. Clearly $\sigma(p) = 1+p$, and by formulas (10) we have $t_2(p) = 2$. Further by Theorem 3

$$(11) \quad pt_2(p) = 2\sigma(p) - 8\sigma(p/2) + \sum_{j=1}^{p-1} t_2(j) \left(2\sigma(p-j) - 8\sigma\left(\frac{p-j}{2}\right) \right).$$

Thus

$$2p = 2(1+p) + 2 \sum_{j=1}^{p-1} t_2(j) \left(\sigma(p-j) - 4\sigma\left(\frac{p-j}{2}\right) \right).$$

\square

Theorem 9. *If $2n+1$ is prime, then*

$$\sum_{j=0}^{n-1} t_4(j) \left(\sigma(n-j) - 4\sigma\left(\frac{n-j}{2}\right) \right) = \frac{n(n+1)}{2} \quad (= \sum_{j=1}^n j).$$

Proof. Assume that $2n+1$ is prime. Then by identities (10) we have $t_4(n) = 2n+2$. Moreover, from Theorem 3 we get

$$(12) \quad nt_4(n) = 4 \sum_{j=0}^{n-1} t_4(j) \left(\sigma(n-j) - 4\sigma\left(\frac{n-j}{2}\right) \right).$$

Combining these two identities proves the result. \square

Theorem 10. *If $4n+3$ is prime, then*

$$\sum_{j=0}^{n-1} t_6(j) \left(\sigma(n-j) - 4\sigma\left(\frac{n-j}{2}\right) \right) = \frac{n(n+1)(2n+1)}{6} \quad (= \sum_{j=1}^n j^2).$$

Proof. If $4n+3$ is prime, then by (10) we have

$$t_6(n) = -\frac{1}{8}(1 - (4n+3)^2) = (n+1)(2n+1).$$

Next by Theorem 3 we have

$$nt_6(n) = 6 \sum_{j=0}^{n-1} t_6(j) \left(\sigma(n-j) - 4\sigma\left(\frac{n-j}{2}\right) \right).$$

Using the previous two formulas we obtain the result. \square

5. FACTS ON SOME INFINITE PRODUCTS

Throughout this section we suppose that x is a complex number such that $|x| < 1$.

Theorem 11. *Let $a, b, n \in \mathbb{N}$ and let I be a nonempty subset of $\{0, 1, 2, \dots, b-2\}$. Then the coefficients of the infinite product*

$$\prod_{n=1}^{\infty} \prod_{i \in I} (1 - x^n)^{-a} (1 - x^{bn-i})^a$$

are all positive.

Proof. Write

$$\prod_{n=1}^{\infty} \prod_{i \in I} (1 - x^n)^{-a} (1 - x^{bn-i})^a = \sum_{n=0}^{\infty} A(n) x^n.$$

We show by induction that $A(n) = 0$ for all $n \in \mathbb{N}_0$. Clearly $A(0) = 1 > 0$. Now assume that the assertion is true for $j = 0, 1, \dots, n-1$. By Theorem 1 we have

$$nA(n) = a\sigma(n) - a \sum_{i \in I} \sigma_{i,b}(n) + \sum_{j=1}^{n-1} A(j) \left(a\sigma(n-j) - a \sum_{i \in I} \sigma_{i,b}(n-j) \right).$$

But

$$a\sigma(n) - a \sum_{i \in I} \sigma_{i,b}(n) = a \sum_{i=0}^{b-1} \sigma_{i,b}(n) - a \sum_{i \in I} \sigma_{i,b}(n) \geq a_{1,b}(n) > 0.$$

Thus by the induction hypothesis we have $nA(n) > 0$ and therefore $A(n) > 0$. \square

For our next result we will need the following lemma.

Lemma 7. [7, Exercise 22, p. 248] *If $n \in \mathbb{N}$, then*

$$R(n) = 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8) > 0.$$

Proof. If $8 \nmid n$, then

$$R(n) = 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) \geq 4\sigma(n) - 4\sigma(n/2) > 0.$$

If $8 \mid n$, say $n = 8k$, then repeatedly application of identities (1) we obtain

$$\begin{aligned} r(8k) &= 4\sigma(8k) - 4\sigma(4k) + 8\sigma(2k) - 32\sigma(k) \\ &= 4\sigma^o(8k) + 4\sigma^o(4k) + 16\sigma^o(2k) \\ &> 0. \end{aligned}$$

This proves the lemma. \square

Theorem 12. *If*

$$\prod_{n=1}^{\infty} (1+x^n)^4 (1+x^{2n})^2 (1+x^{4n})^4 = \sum_{n=0}^{\infty} \alpha(n) x^n,$$

then $\alpha(n) > 0$ for all $n \in \mathbb{N}_0$.

Proof. By induction on n . Clearly $\alpha(0) = 1 > 0$. Suppose now that the statement holds for $j = 0, 1, \dots, n-1$. Note that

$$\prod_{n=1}^{\infty} (1+x^n)^4 (1+x^{2n})^2 (1+x^{4n})^4 = \prod_{n=1}^{\infty} (1-x^n)^{-4} (1-x^{2n})^2 (1-x^{4n})^{-2} (1-x^{8n})^4,$$

which by Theorem 1 leads to

$$\begin{aligned} n\alpha(n) &= 4\sigma(n) - 2\sigma^E(n) + 2\sigma_{0,4}(n) - 4\sigma_{0,8}(n) + \\ &\sum_{j=1}^{n-1} \alpha(j) (4\sigma(n-j) - 2\sigma^E(n-j) + 2\sigma_{0,4}(n-j) - 4\sigma_{0,8}(n-j)). \end{aligned}$$

Further, by (1) and Lemma 7 we get

$$4\sigma(n) - 2\sigma^E(n) + 2\sigma_{0,4}(n) - 4\sigma_{0,8}(n) = 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8) > 0.$$

Then by the induction hypothesis we have $n\alpha(n) > 0$, and thus $\alpha(n) > 0$. \square

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